

Telegraph equations for the case of a waveguide with moving boundary

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Abstract

Telegraph equation describing the compression of electromagnetic waves in a waveguide (resonator) with moving boundary are derived. It is shown that the character of oscillations of the compressed electromagnetic field depends on the parameters of the resonator, and under certain conditions, the oscillations of voltage (current) yield the exponential growth, leading to a noticeable change in the radiation losses.

1 Introduction

The experiments to produce ultrahigh magnetic fields using magnetic cumulative generators (flux compression generators), pioneered by A.D. Sakharov [1] and C.M. Fowler [2], gave rise to a rapidly developing research area related to high energy density physics (see, e.g., [3]). A distinctive feature of these experiments is the use of fast-moving under explosion shells (shock waves)

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to compress the magnetic field. For example, in helical flux compression generators (FCGs), the compressed magnetic field proves trapped between the solenoid and the liner. From the electrodynamical viewpoint, the closed region containing a magnetic (an electromagnetic) field and undergoing compression is a resonator with moving boundary.

The transformation of a microwave field during the compression of a cylindrical resonator was first studied by L.P. Feoktistov and V.V. Klimov in [4]. The authors of [4] derived the explicit expression describing the electromagnetic field inside the resonator in a particular case of axisymmetric mode TM_{010} (E_{001}) of a compressed cylindrical resonator whose walls move at constant velocity and showed that the energy of the field in the resonator increases, because the frequency of the field is increased. Let us note here that the structure of FCG resonators can be more complicated. The helical FCG, for example, is a coaxial resonator whose mode structure differs considerably from that of a cylindrical resonator. In particular, multiply-connected cross section leads to the presence of a TEM wave. The coaxial resonator can be considered as a section of a coaxial waveguide with closed boundaries, and telegraph equations can be applied to analyze oscillations in the resonator. However, commonly-known telegraph equations [5,6] are formulated for the case when the geometric dimensions of the line (coaxial waveguide) are time-independent.

In this paper, we derived telegraph equations to describe the compression of a TEM wave for the case of a waveguide (resonator) with moving boundary. The derived equations are used to analyze how the oscillations of the compressed electromagnetic field depend on the parameters of the resonator.

2 Telegraph equations for moving circuits

Let us consider a circuit moving in the electromagnetic field with a velocity $\vec{v}(t)$. Due to relativistic effects, the current induced in the conductor is [5]

$$\vec{j} = \sigma \left(\vec{E} + [\vec{v}\vec{B}] \right), \quad (1)$$

where \vec{E} is the electric field strength and \vec{B} is the magnetic inductance. It follows from (1) that the electromotive force acting in a moving closed circuit is [5]

$$\varepsilon = \oint_{l(t)} \left(\vec{E} + [\vec{v}(t)\vec{B}] \right) d\vec{l}. \quad (2)$$

According to Faraday's law, the electromotive force (2) equals the time rate of change of magnetic flux Φ :

$$\varepsilon = -\frac{d\Phi}{dt}. \quad (3)$$

Taking the time derivative in (1), we should allow for the total change of the magnetic flux through the circuit (associated with both time change of the magnetic inductance and change in the position of the circuit) [5].

Let us consider the section of length Δx of the coaxial waveguide and the circuit $A B C D$ (see Fig. 1).

Let us introduce the inductance per unit length $L_0(t)$ and capacitance of the unit length $C_0(t)$ of the line. Because the dimensions of the waveguide (the moving line) depend on time, L_0 and C_0 also depend on time. The inductance and capacitance of the waveguide section of length Δx are $L_0\Delta x$ and $C_0\Delta x$,

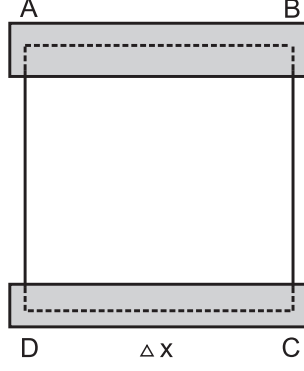


Figure 1. Section of length Δx of the coaxial waveguide and the circuit $A B C D$. respectively, and I is the current. The magnetic flux Φ through the circuit $A B C D$ is

$$\Phi = L_0 \Delta x I.$$

This yields

$$L_0 \frac{\partial I}{\partial t} \Delta x + \frac{\partial L_0}{\partial t} I \Delta x + \varepsilon(x, t) = 0. \quad (4)$$

Let us make use of the equality [6]

$$\varepsilon(x, t) = R_0 \Delta x I + \frac{\partial U}{\partial x} \Delta x, \quad (5)$$

where R_0 is the resistance of the unit length of the line and U is the voltage between the conductors.

As a result, we have

$$L_0(x, t) \frac{\partial I(x, t)}{\partial t} + \frac{\partial L_0(x, t)}{\partial t} I(x, t) + R_0(x, t) I(x, t) + \frac{\partial U(x, t)}{\partial x} = 0. \quad (6)$$

The charge Q of the section having the length Δx is given by $Q = C_0 \Delta x U(x, t)$.

Using the charge density continuity equation, we can obtain the following

equality:

$$C_0 \frac{\partial U}{\partial t} + \frac{\partial C_0}{\partial t} U(x, t) + g_0 U + \frac{\partial U}{\partial x} = 0. \quad (7)$$

Here g_0 , as in well as in the line with time-constant capacitance, describes the leakage current. In a similar manner as in the case of constant parameters of the system [6], we can present the system under consideration in terms of the limit of the set of sequential elements (see Fig. 2).

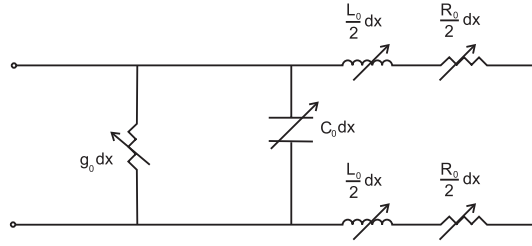


Figure 2. Circuit with time-dependent parameters.

Let us write the derived pair of equations for I and U in a similar form as the standard telegraph equations with time-independent parameters:

$$\begin{aligned} -\frac{\partial U}{\partial x} &= L_0(t, x) \frac{\partial I}{\partial t} + \left(\frac{\partial L_0(t, x)}{\partial t} + R_0(t, x) \right) I \\ -\frac{\partial I}{\partial x} &= C_0(t, x) \frac{\partial U}{\partial t} + \left(\frac{\partial C_0(t, x)}{\partial t} + g_0(t, x) \right) U. \end{aligned} \quad (8)$$

This equation set together with the initial and boundary conditions allows finding the current and voltage in a coaxial resonator (waveguide, line) with moving conductors (with time and coordinate-dependent L_0 , C_0 , R_0 , and g_0).

The derivative $\frac{\partial L_0}{\partial t}$ acts as resistance (in the same manner as in the situation known in the theory of FCG [3]). The derivative $\frac{\partial C_0}{\partial t}$ has an effect on the leakage currents, which can lead to the increase in the current instead of the drop in the case when $\frac{\partial C_0}{\partial t} < 0$.

Now let us assume that the quantities L_0 , C_0 , R_0 , and g_0 are all independent of the coordinate x . By differentiation of (8) with respect to the coordinate x , we can obtain the following second-order equations for voltage and current:

$$\begin{aligned} \frac{\partial^2 U}{\partial x^2} = & L_0 C_0 \frac{\partial^2 U}{\partial t^2} + \left[\frac{\partial L_0 C_0}{\partial t} + L_0 \frac{\partial C_0}{\partial t} + L_0 g_0 + R_0 C_0 \right] \frac{\partial U}{\partial t} \\ & + \left[\frac{\partial}{\partial t} \left(L_0 \frac{\partial C_0}{\partial t} \right) + \frac{\partial L_0 g_0}{\partial t} + R_0 \frac{\partial C_0}{\partial t} + R_0 g_0 \right] U, \end{aligned} \quad (9)$$

$$\begin{aligned} \frac{\partial^2 I}{\partial x^2} = & L_0 C_0 \frac{\partial^2 I}{\partial t^2} + \left[\frac{\partial L_0 C_0}{\partial t} + C_0 \frac{\partial L_0}{\partial t} + g_0 L_0 + R_0 C_0 \right] \frac{\partial I}{\partial t} \\ & + \left[\frac{\partial}{\partial t} \left(C_0 \frac{\partial L_0}{\partial t} \right) + \frac{\partial C_0 R_0}{\partial t} + g_0 \frac{\partial L_0}{\partial t} + g_0 R_0 \right] I, \end{aligned} \quad (10)$$

We shall recall here that $L_0 C_0 = \frac{1}{v^2}$, where v is the wave's phase velocity. Equations (9) and (10) hold true when L_0 , C_0 , R_0 , and g_0 are independent of x . Otherwise we should apply (8).

Let us consider the coaxial resonator (waveguide section, line) with time-independent length l and varying transverse dimensions. Because the coefficients on the right-hand side of equations (9) and (10) are independent of the coordinate x , we can use the separation of variables method. As a result, (9) yields

$$\frac{\partial^2 \varphi}{\partial x^2} + k^2 \varphi = 0. \quad (11)$$

With due account of the boundary conditions, this equation is solvable for a set of eigenwavenumbers k_n and eigenfunctions $\varphi_n(x)$ that are orthogonal and normalized.

By way of example, we shall perform all further transformations using (9) for the voltage $U(x, t)$. Serial expansion of $U(x, t)$ with respect to the functions $\varphi_n(x)$ gives

$$U(x, t) = \sum_n A_n(t) \varphi_n(x). \quad (12)$$

As a result, we have:

$$\frac{\partial^2 A_n}{\partial t^2} + \omega_n^2(t) A_n + b(t) \frac{\partial A_n}{\partial t} + f(t) A_n = 0, \quad (13)$$

where $\omega_n = k_n v(t)$,

$$b(t) = v^2(t) \left[\frac{\partial L_0 C_0}{\partial t} + L_0 \frac{\partial C_0}{\partial t} + L_0 g_0 + R_0 C_0 \right], \quad (14)$$

$$f(t) = v^2(t) \left[\frac{\partial}{\partial t} \left(L_0 \frac{\partial C_0}{\partial t} \right) + \frac{\partial L_0 g_0}{\partial t} + R_0 \frac{\partial C_0}{\partial t} + g_0 R_0 \right]. \quad (15)$$

Thus, we get the equation of the form:

$$\frac{\partial A_n}{\partial t^2} + b(t) \frac{\partial A_n}{\partial t} + \Omega_n^2(t) A_n = 0, \quad (16)$$

where $\Omega_n^2(t) = \omega_n^2(t) + f(t)$.

If the characteristic period of the changes in the system dimensions is long compared to the oscillation period in the coaxial resonator, then we can seek for the solution of (16) in the form

$$A_n(t) = P_{1n}(t) e^{i \int_0^t \Omega_n(t') dt'} + P_{2n}(t) e^{-i \int_0^t \Omega_n(t') dt'}. \quad (17)$$

Let us suppose further that $v^2 = \frac{1}{L_0 C_0}$ is time-independent and the losses are absent ($g_0 = R_0 = 0$).

In this case

$$\Omega_n(t) = \sqrt{\omega_n^2(t) + \frac{1}{L_0 C_0} \frac{\partial}{\partial t} \left(L_0 \frac{\partial C_0}{\partial t} \right)},$$

i.e.,

$$\Omega_n(t) = \sqrt{\omega_n^2(t) + \frac{1}{L_0 C_0} \frac{\partial L_0}{\partial t} \cdot \frac{\partial C_0}{\partial t} + \frac{1}{C_0} \frac{\partial^2 C_0}{\partial t^2}} \quad (18)$$

For an ordinary coaxial waveguide, $L_0 C_0 = \frac{1}{v^2} = \text{const}$, and so if the waveguide is contracted (or expanded), then L_0 drops (increases), whereas C_0 increases (drops). As a consequence, the second term in (18) becomes negative, and the situation occurs when $\Omega_n(t)$ becomes imaginary, the oscillations at the frequency Ω_n vanish, and one of the exponents in (17) starts rising. A similar situation emerges for the current in the waveguide. With reducing longitudinal dimensions of the coaxial resonator, the frequency ω_n builds up due to increasing wavenumber k_n . As a result, the change in geometrical dimensions of a coaxial resonator leads to a significant change in the radiation loss in the system.

Here we discussed a coaxial resonator (waveguide). As is known, to describe different types of resonators in a stationary case, we can introduce inductances and capacitances. In a nonstationary case, we can introduce time-dependent $L(t)$ and $C(t)$ and derive equations similar to those obtained in this paper.

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